South American Journal of Logic
Vol. 5, n. 2, pp. 231–259, 2019
ISSN: 2446-6719

Transreal Proof of the Existence of Universal Possible Worlds

Walter Gomide, Tiago S. dos Reis and James A. D. W. Anderson

Abstract

We adopt an earlier approach of using the set of transreal numbers as a total semantics, supplying classical truth values, dialetheaic values, fuzzy and gap values. We generalise Boolean algebras to trans-Boolean algebras, established on the set of transreal numbers. We give a formal proof that this total, transreal semantics contains classical, fuzzy and paraconsistent semantics by establishing homomorphisms with trans-Boolean algebras. We establish a trans-Cartesian space where the axes are atomic propositions, the co-ordinates are transreal numbers, points are possible worlds, and the set of all points is the set of all possible worlds. We show that this set is a topological metric space. Hence we can measure distances between possible worlds. We generalise vector spaces to transvector spaces so that we can apply geometrical transformations to possible worlds. We define accessibility relations between possible worlds in terms of translinear transformations. Finally we prove the existence of hypercyclic vectors in this transreal space of all possible worlds, which is to say we prove that there are universal possible worlds, any one of which approximates all possible worlds by repeated applications of a single, translinear, actually linear, operator.

Keywords: transreal numbers, logic, total semantics, possible worlds, hypercyclic vectors.

Introduction

We set out to show that there are universal possible worlds which have the topological property of being hypercyclic, which means they can access every world in sequences of worlds that comes arbitrarily close to every possible world. Such universal worlds are theories of everything – everything that can be expressed in written languages - including everything meaningful and everything nonsensical. That there are such worlds, that there are infinitely many of them and that they are spread, infinitely densely, throughout the space of all possible worlds seems, to us, worthy of remark. For example it means that there are many hypercyclic worlds which come arbitrarily close to describing the real world we live in and which, by virtue of being hypercyclic, can describe all possible worlds. This means it is possible that the human mind might embark on a never ending process of discovering everything.

Our method of proof is to establish the space of all possible worlds as a geometrical space and to prove certain algebraic and topological properties of that space. Along the way we find it expedient to generalise various logical and mathematical entities so that they are fit for our purpose. We consider various philosophical issues both in this Introduction and, more fully, in the Discussion.

We begin by considering what set we should use as semantic values, that is as truth values. This is a critical decision because this set will provide the co-ordinate values for our logical space. In [3][16] we proposed using the set of transreal numbers.

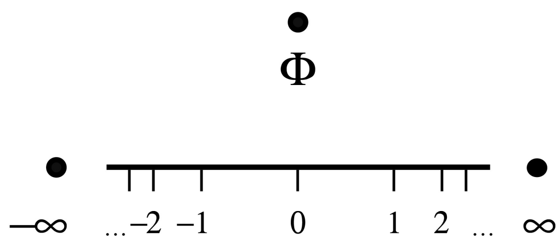

Figure 1: Transreal Number Line

The transreal numbers [5][35] are made up of the real numbers, together with three, definite, non-finite numbers: negative infinity, positive infinity, and nullity. The proof of consistency of the transreal arithmetic can be found in [35]. In Figure 1 the real numbers are shown as a continuous line of some finite length in the figure. The axis is scaled to allow all real numbers to be laid out in the figure. Positive infinity, ∞, lies to the right of the real-number line, but after a space. This space is a necessary and essential property of the transreal numbers [4][34][33]. Similarly negative infinity, $-\infty$, lies to the left of the real-number line, after a space. Nullity, Φ, lies off the real number line. All of the real numbers and both positive and negative infinity are ordered so that negative infinity is the smallest of these numbers and positive infinity is the largest of them. Nullity is not ordered, it is neither small nor large, nor any size in between. Its size is nullity.

In [3][16] negative infinity models the classical truth value False and positive infinity models the classical truth value True. The real numbers (in the range from zero to one) model fuzzy values that describe the extent to which an element is a member of a set [43][18]. The entire set of real numbers models dialeathic values that have degrees of both falsehood and truthfulness [12][31][32]. Negative values are more False than True, positive values are more True than False and zero is equally False and True. Nullity models gap values that are neither False nor True and which, more generally, have no degree of falsehood or truthfulness [39][14][38]. Thus we can model the semantic values of many logics.

We generalise the notion of Boolean logic to trans-Boolean logic and prove that the transreal numbers do model classical, fuzzy and a particular paraconsistent logic by establishing homomorphisms between these logics and trans-Boolean logic.

We note, in passing, that the transreal numbers may be extended by the addition of transfinite ordinals that fall in the space between the reals and infinity and by their reciprocals which fall between zero and all rational numbers, producing a hyperdense part of the line shown in Figure 1. We make no use of these additional numbers here but we note that they are available to support those areas of logic that rely on them.

The idea of logical space is inspired by Wittgenstein's conception that the world's logical form is given by a picture that is a "configuration of objects." See [41][42] sections 2, 3 and, especially 3.4. Thus, just as physical objects are arranged in physical space, so logical objects are arranged in a "logical space" [13]. Wittgenstein did not define precisely his notion of logical space but we define ours here.

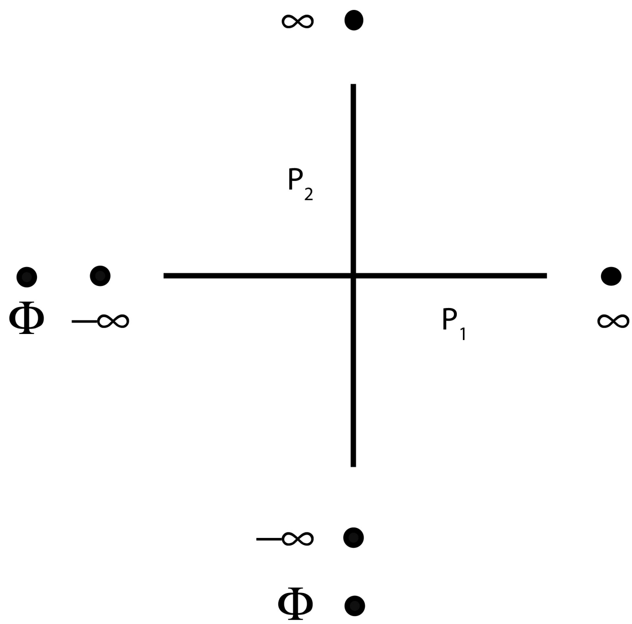

Figure 2: Trans-Cartesian Axes

Figure 2 shows a generalisation of a 2D Cartesian co-ordinate frame. Instead of laying off the axes as real numbers, we lay them off as transreal numbers. Thus the horizontal abscissa can take on any transreal number as value and, entirely independently of this, the vertical ordinate can take on any transreal number. In our topological proofs we use indefinitely many axes where any or all co-ordinates can be nullity and, motivated by Frege's notion of compositionality [15][19], we require that logical connectives, applied to a gap value, produce a gap value as result.

We notionally label each axis with a unique atomic proposition so that a co-ordinate, on a labelled axis, is the degree to which the labelled proposition is True, False or Gap. Thus points in this space are arrangements of semantic values of atomic propositions. In other words, points in this space are possible worlds. Once we have established that our logical space supports a trans-Boolean algebra, we may take the axes or else points to be terms in a trans-Boolean algebra.

The set of all possible worlds has cardinality at least as great as the cardinality of the set that has all combinations of the classical values True and False in each co-ordinate. This cardinality is $2^{\aleph_0}$. The set of points in our transreal space has the cardinality, c, of the real number line. If we take Cantor's continuum hypothesis true then $c = 2^{\aleph_0} = \aleph_1$ so that our logical space has sufficient cardinality to describe the classical set of all possible worlds. In fact, by our construction, there is a one-to-one correspondence between the points in our logical space and the worlds in the set of all possible worlds whose parameters vary over classical, real and transreal values. This establishes the labelling of the points in our logical space.

We then generalise the notion of a vector so that it can operate as a transvector in our trans-Cartesian space. We give a distance metric so that we can talk about the distance between any two possible worlds and can give the vector that passes from one world to another. In the Discussion we show how our geometrical space relates to the system of spheres described by Lewis in his celebrated book *Counterfactuals* [24].

We define accessibility between possible worlds, very generally, as the existence of suitable translinear transformations in our logical space.

We extend the topological notion of hypercyclicity [8][25] so that it holds in our logical space. A hypercyclic vector has indefinitely many ($\aleph_0$) elements. When it is operated on by a certain kind of linear operator, it generates new vectors in a structure called an orbit. The elements of an orbit lie arbitrarily closely to any element in the space and sequences of elements can be chosen, from the orbit, so that they converge, arbitrarily closely, to any element in the space. We use the backward shift operator to generate an orbit of possible worlds from a single, hypercyclic, possible world. The backward shift operator shuffles all of its co-ordinate values down one place so that the first co-ordinate value drops off the beginning of the vector, in a process that is exactly like running Hilbert's hotel paradox backwards. Some sequences of possible worlds then converge arbitrarily closely to any particular possible world and there are so many sequences that every possible world is approached in this way. The proof shows that there are infinitely many (at least $\aleph_0$) hypercyclic, or universal, worlds and that these are spread with infinite density throughout the space of possible worlds.

1 Transreal Numbers

The transreal numbers, pictured in Figure 1, were proposed in order to apply them to computing [1]. This set of numbers, denoted by $\mathbb{R}^T$, is formed by the real numbers and the three novel elements negative infinity, infinity and nullity, which are denoted, respectively, by $-\infty$, ∞ and Φ. Therefore $\mathbb{R}^T = \mathbb{R} \cup \{-\infty, \infty, \Phi\}$. In the set of transreal numbers, division by zero is allowed. Specifically $-1/0 = -\infty$, $1/0 = \infty$ and $0/0 = \Phi$. The arithmetic and order relation defined on $\mathbb{R}^T$ is such that, for each $x, y \in \mathbb{R}^T$, it follows that [5]:

i) If $x \in \mathbb{R}$ then $-\infty < x < \infty$.

ii) If $x \in \mathbb{R}^T$ the following does not hold $x < \Phi$ or $\Phi < x$.

iii) $-(\infty) = -\infty, \quad -(-\infty) = \infty \quad$ and $\quad -\Phi = \Phi$.

iv) $\infty^{-1} = 0, \quad (-\infty)^{-1} = 0, \quad \Phi^{-1} = \Phi \quad$ and $\quad 0^{-1} = \infty$.

v) $\infty + x = \begin{cases} \Phi & , \text{ if } x \in \{-\infty, \Phi\} \\ \infty & , \text{ otherwise} \end{cases}$, $-\infty + x = -(\infty - x)$ and $\Phi + x = \Phi$.

vi) $\infty \times x = \begin{cases} \Phi & , \text{ if } x \in \{0, \Phi\} \\ -\infty & , \text{ if } x < 0 \\ \infty & , \text{ if } x > 0 \end{cases}$, $-\infty \times x = -(\infty \times x)$ and $\Phi \times x = \Phi$.

vii) $x - y = x + (-y)$.

viii) $x \div y = x \times y^{-1}$.

The transreal numbers are an extension of the real numbers and can be made into a topological space[1] with the following properties [4][33]. The open subsets, of the transreal topology, are defined by arbitrarily many unions of finitely many intersections of the following four kinds of interval:

i) (a, b) where $a, b \in \mathbb{R}$,

ii) $[-\infty, b)$ where $b \in \mathbb{R}$,

iii) $(a, \infty]$ where $a \in \mathbb{R}$ and

iv) $\{\Phi\}$.

[1] A set is a *topological space* if and only if it has a topology. Let X be an arbitrary set. As usual let $\mathcal{P}(X)$ be the powerset of X, that is is the set of all subsets of X. Then $\tau \subset \mathcal{P}(X)$ is a *topology on* X if and only if $\emptyset, X \in \tau$, any union of sets in τ belongs to τ and any intersection of finitely many sets in τ belongs to τ. The elements of τ are called *open (sub)sets of* X [27].

Thus $\mathbb{R}^T$ is a topological space. Furthermore $\mathbb{R}^T$ is a Hausdorff[2], disconnected[3], separable[4] and compact[5] space. Notice that Φ is the unique isolated point[6] of $\mathbb{R}^T$. Moreover the topology of $\mathbb{R}^T$ contains the topology of $\mathbb{R}$, that is, when it is restricted to subsets of $\mathbb{R}$, it coincides with the topology of $\mathbb{R}$. In this way the transreal limit is a generalisation of the real limit. Firstly wherever a real number occurs as the limit of a real sequence, that real number occurs identically as the limit of the corresponding transreal sequence. Secondly wherever infinities occur as limit *symbols* in a *divergent,* real sequence, they occur as definite, transreal *numbers* in the corresponding, *convergent*, transreal sequence.

The set of transreal numbers is also a metrisable space[7] with the following metric (among many others): $d : \mathbb{R}^T \times \mathbb{R}^T \to \mathbb{R}$,

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 2, & \text{if } x = \Phi \text{ or else } y = \Phi \\ |\varphi(x) - \varphi(y)|, & \text{otherwise} \end{cases}, \tag{1}$$

where φ is the homeomorphism[8] $\varphi : [-\infty, \infty] \to [-1, 1]$,

$$\varphi(x) = \begin{cases} -1 & , \text{ if } x = -\infty \\ \dfrac{x}{1+|x|} & , \text{ if } x \in \mathbb{R} \\ 1 & , \text{ if } x = \infty \end{cases}.$$

2 Total Semantics

In this section we develop a model for a total semantics. As already stated, the semantics is to contain at least the classical truth values, a contradiction (paraconsistent or dialaethetic) value, fuzzy values and a gap value [3][16].

---

[2] A topological space X is a Haussdorf space if and only if for any distinct $x, y \in X$, there are open sets $U, V \subset X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ [27].

[3] A topological space X is disconnected if and only if there are non-empty, open sets $U, V \subset X$ such that $U \cup V = X$ and $U \cap V = \emptyset$ [27].

[4] A topological space X is said to be separable if and only if it has a dense countable subset. A subset D, of a topological space X, is dense in X if and only if all element of X are elements or limit points of D [27].

[5] A topological space X is said to be compact if and only if, for all classes of open subsets of X, $\{U_\alpha;\ \alpha \in I\}$ (where I is an arbitrary set) such that $X \subset \bigcup_{\alpha \in I} U_\alpha$, there is a finite subset $\{U_{\alpha_k};\ 1 \leq k \leq n\}$ (for some $n \in \mathbb{N}$) of $\{U_\alpha,\ \alpha \in I\}$ such that $X \subset \bigcup_{k=1}^{n} U_{\alpha_k}$ [27].

[6] An element, x, of a topological space X, is said to be an isolated point if and only if there is a neighbourhood $U \subset X$ of x such that $U \cap V = \emptyset$ for all open $V \subset X$ with $V \neq U$.

[7] A topological space is metrisable if and only if it possesses a metric which is compatible with its topology [27].

[8] A homeomorphism is a continuous bijective function whose inverse is continuous [27].

Classical logics are grounded on the principle that each proposition assumes one and only one of the following semantic values: False, True. Thus the set of semantic values is given by $\{F, T\}$ and the connectives are determined by the functions [37] [40]:

$$\neg_{\mathrm{c}} : \{F, T\} \longrightarrow \{F, T\}$$
$$\neg_{\mathrm{c}} F = T, \quad \neg_{\mathrm{c}} T = F$$

$$\vee_{\mathrm{c}} : \{F, T\} \times \{F, T\} \longrightarrow \{F, T\}$$
$$\begin{array}{ll} F \vee_{\mathrm{c}} F = F, & F \vee_{\mathrm{c}} T = T \\ T \vee_{\mathrm{c}} F = T, & T \vee_{\mathrm{c}} T = T \end{array}$$

$$\wedge_{\mathrm{c}} : \{F, T\} \times \{F, T\} \longrightarrow \{F, T\}$$
$$\begin{array}{ll} F \wedge_{\mathrm{c}} F = F, & F \wedge_{\mathrm{c}} T = F \\ T \wedge_{\mathrm{c}} F = F, & T \wedge_{\mathrm{c}} T = T \end{array}$$

The subscript "c" indicates that the connectives are defined in classical logic.

Dialetheism is allowed in paraconsistent logics, that is, one admits the existence of a true proposition whose negation is also true [30] [31]. Paraconsistent logics encompass many calculi. The common property of these calculi is that they do not explode following a contradiction. In classical logic, if we admit a contradiction, as a premise or hypothesis of an inference, then every well formed formula of the language is a theorem [26][11]. Paraconsistent logics interdict such "syntactic explosions."

One of the best known paraconsistent logics is the "logic of paradox," created by the English logician Graham Priest [29]. In this paraconsistent logic, we have a semantic value, δ, which represents a contradiction or dialetheia. That is, δ is the semantic value of a proposition that is both True and False. The simplest form of paraconsistent logic is to consider the set of semantic values $\{F, \delta, T\}$ and the connectives determined by the functions [7]:

$$\neg_{\mathrm{p}} : \{F, \delta, T\} \longrightarrow \{F, \delta, T\}$$
$$\neg_{\mathrm{p}} F = T, \quad \neg_{\mathrm{p}} \delta = \delta, \quad \neg_{\mathrm{p}} T = F$$

$$\vee_{\mathrm{p}} : \{F, \delta, T\} \times \{F, \delta, T\} \longrightarrow \{F, \delta, T\}$$
$$\begin{array}{lll} F \vee_{\mathrm{p}} F = F, & F \vee_{\mathrm{p}} \delta = \delta, & F \vee_{\mathrm{p}} T = T \\ \delta \vee_{\mathrm{p}} F = \delta, & \delta \vee_{\mathrm{p}} \delta = \delta, & \delta \vee_{\mathrm{p}} T = T \\ T \vee_{\mathrm{p}} F = T, & T \vee_{\mathrm{p}} \delta = T, & T \vee_{\mathrm{p}} T = T \end{array}$$

$$\wedge_{\mathrm{p}} : \{F, \delta, T\} \times \{F, \delta, T\} \longrightarrow \{F, \delta, T\}$$
$$\begin{array}{lll} F \wedge_{\mathrm{p}} F = F, & F \wedge_{\mathrm{p}} \delta = F, & F \wedge_{\mathrm{p}} T = F \\ \delta \wedge_{\mathrm{p}} F = F, & \delta \wedge_{\mathrm{p}} \delta = \delta, & \delta \wedge_{\mathrm{p}} T = \delta \\ T \wedge_{\mathrm{p}} F = F, & T \wedge_{\mathrm{p}} \delta = \delta, & T \wedge_{\mathrm{p}} T = T \end{array}$$

The subscript "p" indicates that the connectives are defined in a paraconsistent logic.

In fuzzy logics one admits that propositions can assume degrees of truth and degrees of falsehood. In general these degrees vary continuously. The set of fuzzy semantic values is usually given as the interval of real numbers $[0,1]$ and the connectives are the functions determined by [43]:

$\neg_{\mathrm{f}} : [0,1] \longrightarrow [0,1]$
$$\neg_{\mathrm{f}} x = 1 - x$$

$\vee_{\mathrm{f}} : [0,1] \times [0,1] \longrightarrow [0,1]$
$$x \vee_{\mathrm{f}} y = \max\{x, y\}$$

$\wedge_{\mathrm{f}} : [0,1] \times [0,1] \longrightarrow [0,1]$
$$x \wedge_{\mathrm{f}} y = \min\{x, y\}$$

The subscript "f" indicates that the connectives are defined in a fuzzy logic.

We highlight, further, a semantic value, γ, which corresponds to a "gap." That is, γ is the semantic value of a proposition that is neither true nor false. This value is inspired by the ideas of Quine and Strawson [39][14]. In order to determine the action of the logical connectives at γ, we resort to Frege's principle of compositionality. According to Frege [15], if we admit a whole, in which one of its parts lacks reference, then the whole lacks reference. Following Jansen, Frege's principle of compositionality can be stated this way: "The meaning of a compound expression is a function of the meaning of its parts and the syntactic rule by which they are combined" ([19], p.115). More precisely what we call Frege's principle of compositionality is a particular instance of such a principle. We take it that when an expression has at least one of its parts without any reference then the whole expression lacks reference. In terms of propositions, if one allows a molecular proposition, whatever it may be, in which an atomic proposition is a gap, then the molecular proposition is a gap. Frege's principle is very intuitive – basically it says that a whole must have all of its parts, otherwise it is not a whole, but is nothing. As a consequence of Frege's principle, we have:

$$\neg_{\mathrm{g}}\gamma = \gamma, \quad \gamma \vee_{\mathrm{g}} x = x \vee_{\mathrm{g}} \gamma = \gamma \quad \text{and} \quad \gamma \wedge_{\mathrm{g}} x = x \wedge_{\mathrm{g}} \gamma = \gamma \quad \text{for all } x.$$

Our desire is to get a set of numbers which can serve as the set of semantic values encompassing the classical, dialetheaic, fuzzy and gap values. We shall see that transreal numbers are a good and sufficient candidate for this set.

Definition 2.1 *We call each element of* $\mathbb{R}^T$ *a* semantic value*. Hence* $\mathbb{R}^T$ *is* the set of semantic values*.*

Next we define the functions negation, disjunction and conjunction on $\mathbb{R}^T$.

Definition 2.2 *Let* $\neg$ *denote the function* negation*, given by*

$$\begin{array}{rccc} \neg: & \mathbb{R}^T & \longrightarrow & \mathbb{R}^T \\ & x & \longmapsto & \neg x = -x \end{array}.$$

Let $\vee$ *denote the function* disjunction*, given by*

$$\begin{array}{rccc} \vee: & \mathbb{R}^T \times \mathbb{R}^T & \longrightarrow & \mathbb{R}^T \\ & (x,y) & \longmapsto & x \vee y = \begin{cases} \Phi & , \textit{ if } x = \Phi \textit{ or } y = \Phi \\ \max\{x,y\} & , \textit{ otherwise} \end{cases} \end{array}$$

and let $\wedge$ *denote the function* conjunction*, given by*

$$\begin{array}{rccc} \wedge: & \mathbb{R}^T \times \mathbb{R}^T & \longrightarrow & \mathbb{R}^T \\ & (x,y) & \longmapsto & x \wedge y = \begin{cases} \Phi & , \textit{ if } x = \Phi \textit{ or } y = \Phi \\ \min\{x,y\} & , \textit{ otherwise} \end{cases} \end{array}.$$

We want to show that the above defined structure contains the classical, dialaetheiac, fuzzy and gappy structures defined at the beginning of this section. For this, we need to clarify what we mean by, "one structure contains another."

Definition 2.3 *A* trans-Boolean algebra *is a structure* $(X, \neg, \vee, \wedge, \perp, \top)$*, where* X *is a set,* $\perp, \top \in X$*,* $\neg$ *is a function from* X *to* X *and* $\vee$ *and* $\wedge$ *are functions from* $X \times X$ *to* X *such that the following properties are satisfied: (i) existence of an identity element, (ii) commutativity, (iii) associativity and (iv) distributivity. Thus, for all* $x, y, z \in X$*:*

i) $x \vee \perp = x$ *and* $x \wedge \top = x$.

ii) $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$.

iii) $x \vee (y \vee z) = (x \vee y) \vee z$ *and* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

iv) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ *and* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Note that the structure introduced by Definition 2.2 is a trans-Boolean algebra. That is, $(\mathbb{R}^T, \neg, \vee, \wedge, -\infty, \infty)$ is a trans-Boolean algebra. Note also that $(\{F, T\}, \neg_{\mathrm{c}}, \vee_{\mathrm{c}}, \wedge_{\mathrm{c}}, F, T)$, $(\{F, \delta, T\}, \neg_{\mathrm{p}}, \vee_{\mathrm{p}}, \wedge_{\mathrm{p}}, F, T)$ and $([0,1], \neg_{\mathrm{f}}, \vee_{\mathrm{f}}, \wedge_{\mathrm{f}}, 0, 1)$, mentioned at the beginning of this section, are trans-Boolean algebras.

Definition 2.4 *Let* $(X, \neg_X, \vee_X, \wedge_X, \perp_X, \top_X)$ *and* $(Y, \neg_Y, \vee_Y, \wedge_Y, \perp_Y, \top_Y)$ *be two trans-Boolean algebras and let* f *be a mapping* $f : X \longrightarrow Y$*. We say that* f *is a homomorphism of trans-Boolean algebras if and only if* *i)* $\neg_Y \circ f = f \circ \neg_x$,

ii) $\vee_Y \circ (f \times f) = f \circ \vee_X$,

iii) $\wedge_Y \circ (f \times f) = f \circ \wedge_X$,

iv) $f(\perp_X) = \perp_Y$ *and*

v) $f(\top_X) = \top_Y$.

Theorem 2.5 *There are homomorphisms of trans-Boolean algebras*

i) *from* $(\{F, T\}, \neg_c, \vee_c, \wedge_c, F, T)$ *to* $(\mathbb{R}^T, \neg, \vee, \wedge, -\infty, \infty)$,

ii) *from* $(\{F, \delta, T\}, \neg_p, \vee_p, \wedge_p, F, T)$ *to* $(\mathbb{R}^T, \neg, \vee, \wedge, -\infty, \infty)$ *and*

iii) *from* $([0, 1], \neg_f, \vee_f, \wedge_f, 0, 1)$ *to* $(\mathbb{R}^T, \neg, \vee, \wedge, -\infty, \infty)$.

Proof. Let $f : \{F, \delta, T\} \longrightarrow \mathbb{R}^T$ where $f(F) = -\infty$, $f(\delta) = 0$, $f(T) = \infty$ and $g : [0, 1] \longrightarrow \mathbb{R}^T$ where $g(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ for all $x \in [0, 1]$ (For function tan in $\mathbb{R}^T$ see [2][33]). It is not difficult to see that $f_{|\{F,T\}}$, f and g are, respectively, the required homomorphisms. ■

Remark 2.6 *By the Theorem 2.5, we say that the general trans-Boolean algebra* $(\mathbb{R}^T, \neg, \vee, \wedge, -\infty, \infty)$ *contains the three specific trans-Boolean algebras* $(\{F, T\}, \neg_c, \vee_c, \wedge_c, F, T)$ *of classical logic,* $(\{F, \delta, T\}, \neg_p, \vee_p, \wedge_p, F, T)$ *of Priest's paraconsistent logic and* $([0, 1], \neg_f, \vee_f, \wedge_f, 0, 1)$ *of fuzzy logic. Furthermore,* $\neg\Phi = \Phi$, $\Phi \vee x = x \vee \Phi = \Phi$ *and* $\Phi \wedge x = x \wedge \Phi = \Phi$ *for all* $x \in \mathbb{R}^T$.

Remark 2.6 leads us to the following interpretation of the set of semantic values, $\mathbb{R}^T$:

i) $-\infty$ and ∞ correspond, respectively, to the classical values False and True,

ii) 0 corresponds to the dialetheiac value,

iii) the interval $[-\infty, \infty]$ corresponds to the fuzzy values and

iv) Φ corresponds to the gap value.

3 Possible Worlds

As usual we assume that the set of atomic propositions is a countable set because propositions are written in an enumerable language, which may be a synthetic language or a natural language such as Portuguese or English. We assume, further, that the set of atomic propositions is not finite because we suppose we can name any countable number of elements from the continuum: which is to say we assume that there are $\aleph_0$ atomic propositions. Hence the set of atomic propositions can be written in the form $\{P_i;\ i \in \mathbb{N}\} = \{P_1, P_2, P_3,\ \dots\}$, where $P_i \neq P_j$ whenever $i \neq j$.

The concept of possible worlds is very important in logic. According to Leibniz [23], God has in His mind all worlds that could be created, these are actual in His mind. He chose one of these worlds to be the real world (the best world He could create). According to Leibniz there are laws or statements that are true at every world, these are *necessary* propositions or *Reason's truth,* while some other propositions are true at the real world, but not in all worlds: there is some world at which these *contingent* propositions do not hold. So we have, in Leibniz's approach, a metaphysical basis for interpreting the relation between propositions and possible worlds. For further discussion of Leibniz's conception of possible worlds see [22]. Intuitively a possible world is a binding of a proposition to its semantic values. That is, at a given possible world, each atomic proposition takes on a semantic value in $\mathbb{R}^T$. Thus we can interpret a possible world as a function from $\{P_1, P_2, P_3, \dots\}$ to $\mathbb{R}^T$. But this forms a sequence of elements from $\mathbb{R}^T$. So, denoting the set of the sequences of elements from $\mathbb{R}^T$ by $(\mathbb{R}^T)^{\mathbb{N}}$, we adopt the following definition.

Definition 3.1 *We call each element of* $(\mathbb{R}^T)^{\mathbb{N}}$ *a* possible world*. Hence* $(\mathbb{R}^T)^{\mathbb{N}}$ *is* the set of all possible worlds*.*

In this way each possible world is a point in the space $(\mathbb{R}^T)^{\mathbb{N}}$. Given a possible world $w = (w_i)_{i\in\mathbb{N}} \in (\mathbb{R}^T)^{\mathbb{N}}$, for each $i \in \mathbb{N}$, w_i corresponds to the semantic value of P_i in w.

One question that arises is whether there is any relation between possible worlds? Motivated by Kripke's Modal Logic [20][10], we ask how interactions occur between possible worlds, that is, how do possible worlds communicate with each other? Kripke offers a semantics for Modal Logic, based on the notion of a "modal frame." A modal frame is a pair, $< W, R >$, in which W is a set of possible worlds, and R is a binary relation between possible worlds; this relation is called an "accessibility relation." Intuitively the accessibility relation can be seen as a "path" between worlds. This path is used to define the condition by which a proposition is True or False at a world w. For Kripke's semantics see [21]. We want to propose a mathematical object that plays the role of an accessibility relation from a possible world w to a possible world u. We want this relation to be reflexive and transitive. As we shall see, the existence of a continuous linear transformation T on $(\mathbb{R}^T)^{\mathbb{N}}$, such that $T(w) = u$ is a relation with the desired characteristics. However, linear transformations are usually defined in vector spaces[9], but, due to the structure of the transreal numbers, $(\mathbb{R}^T)^{\mathbb{N}}$ is not a vector space. However, $(\mathbb{R}^T)^{\mathbb{N}}$ has some but not all of the properties of a such space. Motivated by this, we define a *transvector space* as follows.

Definition 3.2 *A nonempty set, V, is called a* transvector space on $\mathbb{R}^T$ *if and only if there are two operations $+ : V \times V \longrightarrow V$ and $\cdot : \mathbb{R}^T \times V \longrightarrow V$ (named, respectively, addition and scalar multiplication), such that the following properties are satisfied: additive commutativity, additive associativity, scalar multiplicative associativity, additive identity and scalar multiplicative identity. Which are, respectively, for any $w, u, v \in V$ and $x, y \in \mathbb{R}^T$:*

i) $w + u = u + w$.

ii) $w + (u + v) = (w + u) + v$.

iii) $x \cdot (y \cdot w) = (xy) \cdot w$.

iv) there is $o \in V$ such that $o + w = w$.

v) $1 \cdot w = w$.

The elements of V are called transvectors. *Further $x \cdot w$ is customarily denoted as $w \cdot x$, xw or wx and o as 0.*

Let $w, u \in (\mathbb{R}^T)^{\mathbb{N}}$, where $w = (w_j)_{j\in\mathbb{N}}$ and $u = (u_j)_{j\in\mathbb{N}}$, and $x \in \mathbb{R}^T$. We define $w + u := (w_j + u_j)_{j\in\mathbb{N}}$ and $xw := (xw_j)_{j\in\mathbb{N}}$ and we denote $(0, 0, 0, \dots) \in (\mathbb{R}^T)^{\mathbb{N}}$ simply by 0. We are conscious that we abuse notation in the above definition. However, the reader will have no difficulty in understanding that, in $w + u := (w_j + u_j)_{j\in\mathbb{N}}$, the symbol, $+$, on the left hand side of the equality refers to the addition which is being defined on $(\mathbb{R}^T)^{\mathbb{N}}$ and the symbol, $+$, on the right hand side of the equality refers to addition on $\mathbb{R}^T$ which addition has a prior definition. Analogously for $xw := (xw_j)_{j\in\mathbb{N}}$.

Notice that, with these operations, $(\mathbb{R}^T)^{\mathbb{N}}$ is a transvector space on $\mathbb{R}^T$.

[9] A nonempty set, V, is called a *vector space on a field, F,* if and only if there are two operations $+ : V \times V \longrightarrow V$ and $\cdot : F \times V$ such that, for any $w, u, v \in V$ and $x, y \in F$: (Additive commutativity) $w + u = u + w$, (Additive associativity) $w + (u + v) = (w + u) + v$, (Distributivity of scalar multiplication with respect to vector addition) $x \cdot (w+u) = x \cdot w + x \cdot u$, (Distributivity of scalar multiplication with respect to field addition) $(x+y) \cdot w = x \cdot w + y \cdot w$, (Scalar multiplicative associativity) $x \cdot (y \cdot w) = (xy) \cdot w$, (Additive identity) there is $o \in V$ such that $o + w = w$, (Additive inverse) for all $w \in V$ there is $-w \in V$ such that $w + (-w) = o$ and (Scalar multiplicative identity) $1 \cdot w = w$ [28].

Definition 3.3 *Let V and W be transvector spaces on $\mathbb{R}^T$. We say that $T : V \to W$ is a* translinear *transformation on V if and only if for all $w, u \in V$ and $x \in \mathbb{R}^T$,*

i) $T(w + u) = T(w) + T(u)$ *and*

ii) $T(xw) = xT(w)$.

Definition 3.4 *Given two arbitrary possible worlds $w, u \in (\mathbb{R}^T)^{\mathbb{N}}$, we say that $T : (\mathbb{R}^T)^{\mathbb{N}} \longrightarrow (\mathbb{R}^T)^{\mathbb{N}}$ is a* communication from w to u *if and only if T is a continuous, translinear transformation and satisfies:*

i) every constant sequence is a fixed point of T, in other words, for each $v = (v_i)_{i \in \mathbb{N}} \in (\mathbb{R}^T)^{\mathbb{N}}$ such that $v_i = v_1$ for all $i \in \mathbb{N}$, $T(v) = v$ and

ii) $T(w) = u$.

Definition 3.5 *Given two arbitrary possible worlds $w, u \in (\mathbb{R}^T)^{\mathbb{N}}$, we say that wRu if and only if there is a communication from w to u. We call the relation R an* accessibility relation. *We say that w* accesses *u or that u* is accessible from *w if and only if wRu.*

Proposition 3.6 *The accessibility relation, R, satisfies the reflexive and transitive properties. That is, for all $w, u, v \in (\mathbb{R}^T)^{\mathbb{N}}$, respectively,*

i) wRw.

ii) if wRu and uRv then wRv.

Proof. Let there be arbitrary $w, u, v \in (\mathbb{R}^T)^{\mathbb{N}}$. The identity function, Id, is a communication from w to w, hence wRw. Now suppose that wRu and uRv, that is, there is a communication, T, from w to u and a communication, S, from u to v. So $T, S : (\mathbb{R}^T)^{\mathbb{N}} \longrightarrow (\mathbb{R}^T)^{\mathbb{N}}$ are continuous, translinear transformations, such that every constant sequence is a fixed point of T and S and, furthermore, $T(w) = u$ and $S(u) = v$. Since the composition of continuous, translinear transformations is a continuous, translinear transformation, $S \circ T$ is a continuous, translinear transformation and, furthermore, every constant sequence is a fixed point of $S \circ T$ and $(S \circ T)(w) = S(T(w)) = S(u) = v$. Thus $S \circ T$ is a communication from w to v, whence wRv. ■

3.1 Possible Worlds Which Accesses Any Other By Approximation

The set of all possible worlds $(\mathbb{R}^T)^{\mathbb{N}}$ is a topological space, where the topology is the product topology. That is to say, the set $U \subset (\mathbb{R}^T)^{\mathbb{N}}$ is open if and only if $U = \prod_{j \in \mathbb{N}} A_j$, where A_j is open on $\mathbb{R}^T$ for all $j \in \mathbb{N}$ and $A_j = \mathbb{R}^T$, except for a finite numbers of indexes j [27].

Remark 3.7 *Notice that, since $\mathbb{R}^T$ is a Hausdorff and separable space then $(\mathbb{R}^T)^{\mathbb{N}}$ is also a Hausdorff and separable space. Note also that, since $\mathbb{R}^T$ is compact, by Tychonoff Theorem ([27], Theorem 37.3), $(\mathbb{R}^T)^{\mathbb{N}}$ is compact.*

A topological space is a space that allows us to speak of neighbourhoods, proximity and convergence. In a topological space we can give an exact sense to the notion of "being close to." In a topological space, E, a neighbourhood of a point, $a \in E$, is an open set which contains a. Saying that for any neighbourhood, U of a, there is some $b \in U$, with b different from a, means that one can get a point, b, as close to a as one wants. According to our model of logical space, possible worlds are points in the topological space $(\mathbb{R}^T)^{\mathbb{N}}$. This allows us to speak of proximity and convergence with respect to possible worlds. That is, we have an exact meaning for "a possible world is close to another" and for "a succession of possible worlds converges to a determinate one" when we use the usual definition of convergence of sequences in topological spaces. That is to say, a sequence $(w^{(n)})_{n \in \mathbb{N}} \subset (\mathbb{R}^T)^{\mathbb{N}}$ converges to $w \in (\mathbb{R}^T)^{\mathbb{N}}$ or w is the limit of $(w^{(n)})_{n \in \mathbb{N}}$ or $\lim_{n \to \infty} w^{(n)} = w$ if and only if for each $U \subset (\mathbb{R}^T)^{\mathbb{N}}$, being a neighbourhood of w, there is $n_U \in \mathbb{N}$ such that $w^{(n)} \in U$ for all $n \geq n_U$. Notice that, since $(\mathbb{R}^T)^{\mathbb{N}}$ is a Hausdorff space, the limit of a sequence, if it exists, is unique [27].

Now we introduce the concept of an hypercyclic operator, as developed in functional analysis. The first examples of hypercyclic operators were obtained, in the first half of the twentieth century, by G. Birkhoff [8] and G. MacLane [25]. Since then hypercyclic operators have been developed to a considerable degree. See [17] for a recent review. We use a particular hypercyclic operator, the backward shift, to interpret our spatial model of all possible worlds.

An operator is a function whose domain and counter-domain are the same set. Formally an operator, T, is hypercyclic if there is an element, x, of the domain, such that every other element, of the domain, can be approximated by recursive applications of T on x. That is to say, from just the operator, T, and an hypercyclic element, x, related to T, we can obtain all others elements of the domain by topological approximation. In other words we can come arbitrarily closely to every possible world by recursively applying one operator on any chosen hypercyclic world.

Hypercyclic operators are usually defined on topological vector spaces[10] but the idea of an hypercyclic operator makes sense in arbitrary topological spaces so we can define hypercyclic operators on $(\mathbb{R}^T)^{\mathbb{N}}$.

Given a set X and a function $f : X \longrightarrow X$, we define the *iterates* of f as

$$f^0 = \mathrm{Id}_X,\ f^1 = f,\ f^2 = f \circ f,\ f^3 = f \circ f^2,\ \dots\ ;$$

where Id_X the identity function on X. Also, for each $x \in X$, we define the *orbit of x related to f* as

$$\mathrm{orb}(x, f) := \{x, f(x), f^2(x), \dots\}.$$

Definition 3.8 *Let X be a topological space. A continuous operator*[11]*, T on X, is said to be* hypercyclic *if and only if there is an $x \in X$ such that* $\mathrm{orb}(x, T)$ *is dense in X. In this case x is called an* hypercyclic element *of T. The set of hypercyclic elements of T is denoted as $HC(T)$.*

Remark 3.9 (Hypercyclic worlds come close to all worlds) *The fact that* $\mathrm{orb}(x, T)$ *is dense in X means, in our model of logical space, that a hypercyclic world generates a sequence of worlds such that every world is approached, arbitrarily closely, by worlds in that sequence.*

Now we will see that there exists an hypercyclic operator on $(\mathbb{R}^T)^{\mathbb{N}}$ and, next, we use this operator to interpret our model of possible worlds. Let $B : (\mathbb{R}^T)^{\mathbb{N}} \longrightarrow (\mathbb{R}^T)^{\mathbb{N}}$, $B(w_1, w_2, w_3, \dots) = (w_2, w_3, w_4, \dots)$. The operator B is called a *backward shift.* Notice that B is a translinear transformation on $(\mathbb{R}^T)^{\mathbb{N}}$. We will see that B is an hypercyclic operator on $(\mathbb{R}^T)^{\mathbb{N}}$. Bonet and Peris give a proof that the operator B is hypercyclic on the Fréchet space $\mathbb{K}^{\mathbb{N}}$ [9]. We just adapt that proof to the space $(\mathbb{R}^T)^{\mathbb{N}}$.

Proposition 3.10 *The operator B is continuous in $(\mathbb{R}^T)^{\mathbb{N}}$.*

Proof. Let $U \subset (\mathbb{R}^T)^{\mathbb{N}}$ be an arbitrary open set. We have that $U = A_1 \times A_2 \times A_3 \times \cdots$, where A_j is open on $\mathbb{R}^T$ for all $j \in \mathbb{N}$ and $A_j = \mathbb{R}^T$, except for

[10] A topological vector space is a vector space endowed with a topology such that vector addition and scalar multiplication are continuous functions and every singleton set is closed [36].

[11] If X and Y are topological spaces, the function $f : X \to Y$ is said to be continuous in $x \in X$ if and only if, given a neighborhood V of $f(x)$ in Y, there is a neighborhood U of x in X such that $f(U) \subset V$. A function $f : X \to Y$ is said to be continuous in X if and only if f is continuous in x, for all $x \in X$ [27].

a finite number of indexes j. Notice that $B^{-1}(U) = B^{-1}(A_1 \times A_2 \times A_3 \times \cdots) = \mathbb{R}^T \times A_1 \times A_2 \times \cdots$. In fact,

$$\begin{aligned}
&(w_1, w_2, w_3, \dots) \in B^{-1}(A_1 \times A_2 \times A_3 \times \cdots) && \Leftrightarrow \\
&(w_2, w_3, w_4, \dots) = B(w_1, w_2, w_3, \dots) \in A_1 \times A_2 \times A_3 \times \cdots && \Leftrightarrow \\
&(w_1, w_2, w_3, \dots) \in \mathbb{R}^T \times A_1 \times A_2 \times \cdots .
\end{aligned}$$

Clearly $\mathbb{R}^T \times A_1 \times A_2 \times \cdots$ is open. Since the pre-images of open sets in B are open, B is continuous ([27], Theorem 18.1). ■

Lemma 3.11 (Birkhoff Transitivity Theorem) *Let X be a separable, complete metric space and let $f : X \longrightarrow X$ be a continuous function. If f is topologically transitive, in the following sense: for any pair U, V of non-empty, open subsets of X, there is $n \in \{0\} \cup \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$, then f is hypercyclic. And in this case, the set of hypercyclic elements is a dense set in X.*

Proof. Suppose that f is topologically transitive. Since X has a countable, dense subset, the topology of X has a countable base. Let $(U_k)_{k \in \mathbb{N}}$ be the enumeration of this base. Then x has a dense orbit if and only if, for each $k \in \mathbb{N}$, there is an $n \in \{0\} \cup \mathbb{N}$, such that $f^n(x) \in U_k$. That is to say the set, D, of points whose orbit is dense, is given by

$$D = \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{\infty} (f^n)^{-1}(U_k).$$

Since f is continuous, for each $k \in \mathbb{N}$, $\bigcup_{n=0}^{\infty} (f^n)^{-1}(U_k)$ is open in X. Furthermore each one of these sets is dense in X. In fact if V is a non-empty, open subset of X then, by the hypothesised transitivity of f, we have that $(f^n)^{-1}(U_k) \cap V \neq \emptyset$, for some $n \in \{0\} \cup \mathbb{N}$, whence $\bigcup_{n=0}^{\infty} (f^n)^{-1}(U_k) \cap V \neq \emptyset$. Since X is a Haussdorf, compact space, by the Baire Theorem ([36], 1991; Theorem 2.2), D is dense and, in particular, non-empty. ■

Lemma 3.12 *Let X, Y be topological spaces and let $f : X \longrightarrow X$, $g : Y \longrightarrow Y$ be continuous functions. If g is hypercyclic and there is a continuous function, $\phi : Y \longrightarrow X$, with dense image, such that $f \circ \phi = \phi \circ g$, then f is also hypercyclic.*

Proof. Let there be an element, $y \in Y$, that has a dense orbit related to g. If U is a non-empty, open subset of X then $\phi^{-1}(U)$ is open and is a non-empty subset of Y, because ϕ is continuous and has a dense image. Thus there is $n \in \{0\} \cup \mathbb{N}$ such that $g^n(y) \in \phi^{-1}(U)$. Since $f(\phi(y)) = \phi(g(y))$, one verifies, by induction, that $f^n(\phi(y)) = \phi(g^n(y))$. Thus $f^n(\phi(y)) = \phi(g^n(y)) \in U$. Thus $\phi(y)$ has a dense orbit related to f. ■

Let X, Y be any sets and let $f : X \longrightarrow X$, $g : Y \longrightarrow Y$ be functions. The function $f \times g : X \times Y \longrightarrow X \times Y$ is defined as

$$(f \times g)(x, y) = (f(x), g(y)).$$

Clearly $(f \times g)^n = f^n \times g^n$, for all $n \in \{0\} \cup \mathbb{N}$. Furthermore if X, Y are topological spaces and f, g are continuous then $f \times g$ is also continuous in the product topology of $X \times Y$.

Theorem 3.13 *The operator B is hypercyclic in $(\mathbb{R}^T)^{\mathbb{N}}$.*

Proof. For each $n \in \mathbb{N}$, denote $e^{(n)} = (\underbrace{0, 0, \dots, 0}_{n-1 \text{ coordinates}}, 1, 0, 0, \dots)$, that is $e^{(n)} = (w_j)_{j \in \mathbb{N}}$, where $w_n = 1$ and $w_j = 0$ for all $j \in \mathbb{N} \setminus \{n\}$. Notice that $\lim\limits_{n\to\infty} e^{(n)} = 0$. In fact, let $U \subset (\mathbb{R}^T)^{\mathbb{N}}$ be a neighbourhood of 0. We have that $U = A_1 \times A_2 \times A_3 \times \cdots$, where A_j is a neighbourhood of 0 in $\mathbb{R}^T$ for all $j \in \mathbb{N}$ and there is $n_U \in \mathbb{N}$ such that $A_j = \mathbb{R}^T$ for all $j \geq n_U$. Thus $e^{(n)} \in U$ for all $n \geq n_U$.

Now denote, as c_{00}^T, the set of all sequences of transreal numbers such that only a finite number of coordinates are not equal to zero. Notice that c_{00}^T is dense in $(\mathbb{R}^T)^{\mathbb{N}}$. In fact, for each arbitrary $(w_j)_{j \in \mathbb{N}} \in (\mathbb{R}^T)^{\mathbb{N}}$, we take a sequence $\left(\sum\limits_{j=1}^{n} w_j e^{(j)}\right)_{j \in \mathbb{N}} \subset c_{00}^T$ and we know that $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} w_j e^{(j)} = (w_j)_{j \in \mathbb{N}}$.

Let be $F : c_{00}^T \to c_{00}^T$, where $F(w_1, w_2, w_3, \dots) = (0, w_1, w_2, \dots)$. Notice that

$$B^n(F^n(u)) = u, \text{ for all } u \in c_{00}^T \text{ and for all } n \in \mathbb{N}, \tag{2}$$

and

$$\lim_{n\to\infty} B^n(w) = 0, \text{ for all } w \in c_{00}^T. \tag{3}$$

Furthermore if $u = (u_1, u_2, u_3, \dots) \in c_{00}^T$ then there is $k \in \mathbb{N}$ such that $u_j = 0$ for all $j > k$, whence $u = u_1 e^{(1)} + \cdots + u_k e^{(k)}$ and, so, $F^n(u) = F^n(u_1 e^{(1)} + \cdots + u_k e^{(k)}) = F^n(u_1 e^{(1)}) + \cdots + F^n(u_k e^{(k)}) = u_1 e^{(1+n)} + \cdots + u_k e^{(k+n)}$. Since $\lim\limits_{n\to\infty} e^{(n)} = 0$, we have that $\lim\limits_{n\to\infty} F^n(u) = \lim\limits_{n\to\infty} (u_1 e^{(1+n)} + \cdots + \lim\limits_{n\to\infty} u_k e^{(k+n)}) =$

$\lim_{n\to\infty}(u_1 e^{(1+n)}) + \cdots + \lim_{n\to\infty}(u_k e^{(k+n)}) = u_1 \lim_{n\to\infty} e^{(1+n)} + \cdots + u_k \lim_{n\to\infty} e^{(k+n)} = u_1 \cdot 0 + \cdots + u_k \cdot 0 = 0$. Thus

$$\lim_{n\to\infty} F^n(u) = 0, \text{ for all } u \in c_{00}^T. \tag{4}$$

Now let $W_1, W_2, U_1, U_2 \subset (\mathbb{R}^T)^{\mathbb{N}}$ be arbitrary and non-empty, open sets. Since c_{00}^T is dense in $(\mathbb{R}^T)^{\mathbb{N}}$, there are $w^{(1)} \in W_1 \cap c_{00}^T$, $w^{(2)} \in W_2 \cap c_{00}^T$, $u^{(1)} \in U_1 \cap c_{00}^T$ and $u^{(2)} \in U_2 \cap c_{00}^T$. It follows from (2) and (3) that

$$\lim_{n\to\infty} B^n(w^{(1)}+F^n(u^{(1)})) = \lim_{n\to\infty} B^n(w^{(1)})+\lim_{n\to\infty} B^n(F^n(u^{(1)})) = 0+u^{(1)} = u^{(1)}.$$

And it follows from (4) that

$$\lim_{n\to\infty}(w^{(1)} + F^n(u^{(1)})) = w^{(1)} + 0 = w^{(1)}.$$

Thus there is $n_1 \in \mathbb{N}$ such that $w^{(1)}+F^n(u^{(1)}) \in W_1$ and $B^n(w^{(1)}+F^n u^{(1)}) \in U_1$ for all $n \geq n_1$. In the same way there is $n_2 \in \mathbb{N}$ such that $w^{(2)} + F^n u^{(2)} \in W_2$ and $B^n(w^{(2)} + F^n u^{(2)}) \in U_2$ for all $n \geq n_2$. By choosing $n_0 = \max\{n_1, n_2\}$ we see that $(T \times T)^{n_0}(W_1 \times W_2) \cap (U_1 \times U_2) \neq \emptyset$. Thus $B \times B$ is topologically transitive, whence, by Lemma 3.11, $B \times B$ is hypercyclic in $(\mathbb{R}^T)^{\mathbb{N}} \times (\mathbb{R}^T)^{\mathbb{N}}$.

Clearly the function $\phi : (\mathbb{R}^T)^{\mathbb{N}} \times (\mathbb{R}^T)^{\mathbb{N}} \to (\mathbb{R}^T)^{\mathbb{N}}$, $\phi(x, y) = x$ satisfies the conditions of Lemma 3.12 for $B \times B$ and B. Thus B is hypercyclic. ■

Remark 3.14 (Many hypercyclic possible worlds) *As $HC(B)$ is dense, there are many hypercyclic, possible worlds in our model of logical space.*

Since B is an hypercyclic operator, there is an hypercyclic transvector, w, with respect to B. Since B is a continuous, translinear transformation on $(\mathbb{R}^T)^{\mathbb{N}}$ such that every constant sequence is a fixed point of B, w accesses every element in $\mathrm{orb}(w, B)$. Since B is hypercyclic, $\mathrm{orb}(w, B)$ is dense in $(\mathbb{R}^T)^{\mathbb{N}}$. This means that, given an arbitrary possible world, u, there is a sequence of elements from $\mathrm{orb}(w, B)$ which converges to u. That is, there is a possible world w, which accesses a sequence of possible worlds that converges to u, whatever the possible world u.

4 Distance Between Possible Worlds

In addition to being a topological space, $(\mathbb{R}^T)^{\mathbb{N}}$ is also a metric space[12], with compatible metric and topology. In a metric space we can speak of the distance

[12] A metric space is a set M endowed with a function $d : M \times M \longrightarrow \mathbb{R}$ such that, for any $x, y, z \in M$, $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ [27].

between elements. Hence we can speak of the distance between possible worlds. And, as the metric is compatible with the topology, the concepts previously mentioned – neighbourhoods, proximity and convergence – are all respected by the metric. In [27] there is a simple construction of a metric on $\mathbb{R}^{\mathbb{N}}$ compatible with its topology. Below we adapt that construction to the space $(\mathbb{R}^T)^{\mathbb{N}}$.

Proposition 4.1 *$(\mathbb{R}^T)^{\mathbb{N}}$ is metrisable. More specifically, for each $w, u \in (\mathbb{R}^T)^{\mathbb{N}}$, denote $w = (w_j)_{j\in\mathbb{N}}$ and $u = (u_j)_{j\in\mathbb{N}}$, and let $D : (\mathbb{R}^T)^{\mathbb{N}} \times (\mathbb{R}^T)^{\mathbb{N}} \to \mathbb{R}$ be defined as*

$$D(w,u) = \sup_{j\in\mathbb{N}} \left\{ \frac{d(w_j,u_j)}{j} \right\},$$

where d is defined in (1). We have that D is a metric on $(\mathbb{R}^T)^{\mathbb{N}}$ which induces the topology of $(\mathbb{R}^T)^{\mathbb{N}}$.

Proof. Firstly, let us see that D is, in fact, a metric on $(\mathbb{R}^T)^{\mathbb{N}}$. Clearly, for all $w, u \in (\mathbb{R}^T)^{\mathbb{N}}$: $D(w,u) = 0$ if and only if $w = u$; $D(w,u) = D(u,w)$; and $D(w,u) \geq 0$. If $w, u, v \in (\mathbb{R}^T)^{\mathbb{N}}$ then, for all $j \in \mathbb{N}$, $\dfrac{d(w_j,v_j)}{j} \leq \dfrac{d(w_j,u_j)}{j} + \dfrac{d(u_j,v_j)}{j} \leq \sup_{j\in\mathbb{N}} \left\{ \dfrac{d(w_j,u_j)}{j} \right\} + \sup_{j\in\mathbb{N}} \left\{ \dfrac{d(u_j,v_j)}{j} \right\} = D(w,u) + D(u,v)$ whence $D(w,u) + D(u,v)$ is an upper bound of $\left\{ \dfrac{d(u_j,v_j)}{j};\ j \in \mathbb{N} \right\}$. Thus $D(w,v) = \sup_{j\in\mathbb{N}} \left\{ \dfrac{d(w_j,v_j)}{j} \right\} \leq D(w,u) + D(u,v)$.

Now let us see that the topology induced by D and the product topology of $(\mathbb{R}^T)^{\mathbb{N}}$ are the same. Recall that, in metric spaces, $B(w,r)$ denotes the ball of centre w and radius r, because of this, $B(w,r) = \{u \in (\mathbb{R}^T)^{\mathbb{N}};\ D(u,w) < r\}$ and $B(w_j,r) = \{u_j \in \mathbb{R}^T;\ D(u_j,w_j) < r\}$. Let us see that every open set in the product topology is also open in the topology induced by D. Let U be an arbitrary, open set in the product topology and let $w = (w_j)_{j\in\mathbb{N}} \in U$. We have that $U = \prod_{j\in\mathbb{N}} A_j$, where there is $n \in \mathbb{N}$ such that A_j is open on $\mathbb{R}^T$ for all $j \in \{1,\dots,n\}$ and $A_j = \mathbb{R}^T$ for all $j > n$. Further, $w_j \in A_j$ for all $j \in \{1,\dots,n\}$. For each $j \in \{1,\dots,n\}$, if $w_j = \Phi$ we take $r_j = 1$, hence $B(w_j,r_j) = \{\Phi\} \subset A_j$, if $w_j \neq \Phi$, since φ^{-1} is continuous (φ is the homeomorphism which appears in (1)), there is a positive $r_j \in \mathbb{R}$ such that $B(w_j,r_j) \subset A_j$. Now let $r := \min\left\{ \dfrac{r_j}{j};\ j \in \{1,\dots,n\} \right\}$. If $u = (u_j)_{j\in\mathbb{N}} \in B(w,r)$ then $\dfrac{d(w_j,u_j)}{j} \leq D(w,u) < r$ for all $j \in \mathbb{N}$. For each $j \in \{1,\dots,n\}$, $r \leq \dfrac{r_j}{j}$, whence $\dfrac{d(w_j,u_j)}{j}) < r \leq \dfrac{r_j}{j}$. Hence $d(w_j,u_j) < r_j$, whence $u_j \in$

$B(w_j, r_j) \subset A_j$ so that $u_j \in A_j$ for all $j \in \{1, \ldots, n\}$. Thus $u \in U$, whence $B(w, r) \subset U$.

Now let us see that every ball in metric D is an open set in the product topology. Let there be an arbitrary $w = (w_j)_{j \in \mathbb{N}} \in (\mathbb{R}^T)^{\mathbb{N}}$ and an arbitrary, positive $r \in \mathbb{R}$. Let $n \in \mathbb{N}$ such that $n > \frac{2}{r}$, whence $\frac{2}{n} < r$. For each $j \in \{1, \ldots, n\}$, if $w_j = \Phi$, we take $A_j = \{\Phi\}$ and so $A_j \subset B(w_j, r)$; if $w_j \neq \Phi$, since φ is continuous, there is a neighbourhood A_j of w_j such that $A_j \subset B(w_j, r)$. Let $U = A_1 \times \cdots \times A_n \times \mathbb{R}^T \times \mathbb{R}^T \times \cdots$. If $u = (u_j)_{j \in \mathbb{N}} \in (\mathbb{R}^T)^{\mathbb{N}}$ then $\frac{d(w_j, u_j)}{j} \leq \frac{2}{n}$ for all $j \geq n$. Hence $D(w, u) \leq \max\left\{\frac{d(w_1, u_1)}{1}, \ldots, \frac{d(w_n, u_n)}{n}, \frac{2}{n}\right\}$. Therefore if $u \in U$ then $D(w, u) \leq \max\left\{\frac{d(w_1, u_1)}{1}, \ldots, \frac{d(w_n, u_n)}{n}, \frac{2}{n}\right\} < r$. Thus $u \in B(w, r)$, whence $U \subset B(w, r)$. ∎

As $(\mathbb{R}^T)^{\mathbb{N}}$ is metrical, all results on metric spaces hold.

Corollary 4.2 $(\mathbb{R}^T)^{\mathbb{N}}$ *is a complete,*[13] *metric space.*

Proof. Every compact, metric space is complete and $(\mathbb{R}^T)^{\mathbb{N}}$ is compact and metric. ∎

As $(\mathbb{R}^T)^{\mathbb{N}}$ is complete, all results on complete, metric spaces hold.

Our proof of the existence of hypercyclic, possible worlds is given in terms of a product topology, in which one measures distance over an arbitrarily large but finite number of co-ordinates. In this topology we cannot require proximity of an actually infinite number of co-ordinates but we can require proximity of a potentially infinite number of co-ordinates. But our logical space of all possible worlds, $(\mathbb{R}^T)^{\mathbb{N}}$, has a metric, D, compatible with its topology. Hence we can speak of proximity in terms of distance over an actually infinite number of co-ordinates. See Proposition 4.1, $D(w, u) = \sup_{j \in \mathbb{N}}\left\{\frac{d(w_j, u_j)}{j}\right\}$. Thus D works on all co-ordinates. One might ask how does a metric, which works over infinitely many co-ordinates, agree with a topology which works over only finitely many co-ordinates? The answer is that as j is in the denominator of the metric D, when j is large, the j-th co-ordinate influences the metric, D, to an asymptotically small degree. The proof of Proposition 4.1 gives further details.

[13] A metric space, M, is called complete, if and only if every Cauchy sequence in M converges in M. A sequence $(x_n)_{n \in \mathbb{N}}$, in a metric space, is a Cauchy sequence if and only if, given arbitrary $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

In summary we can interpret the existence of an hypercyclic world in $(\mathbb{R}^T)^{\mathbb{N}}$ as follows: there is a possible world, w, such that, given any possible world u, there is a possible world, v, which is metrically as close to u as one can want, such that w accesses v. This means that w accesses any possible world "by metrical approximation."

5 Discussion

The main achievement of this paper is to prove the existence of universal possible worlds that describe all possible worlds. We regard each universal worlds as a theory of everything. Firstly a universal world is a theory in the sense that it provides a description. Secondly a universal world is a theory of everything in the sense that operating on it produces descriptions of every possible world in our logical universe.

We began by constructing a Cartesian co-ordinate frame. We constructed a countable infinitude, $\aleph_0$, of axes and a continuum, c, of points in space. We chose to label the axes with atomic predicates. This agrees with the usual assumption that predicates are enunciated in a discrete language so that there are countably many predicates. It follows that there are infinitely many atomic predicates because, in some particular logic, we may take finitely many atomic predicates, $p_1, p_2, ..., p_n$ as axioms but we may then take all of the $p_{n+1}, p_{n+2}, ...$ as an infinitude of atomic predicates such that predicate p_{n+i} means "I am the $(n+i)$'th predicate." We require an infinitude of predicates because hypercyclicity is a property only of infinite dimensional spaces [6]. Every point in our Cartesian space is then a possible world whose co-ordinates are the semantic values of its atomic predicates. Equivalently the points are molecular predicates. At this juncture we could have used the usual results of analysis to prove the existence of hypercyclic, possible worlds but this would have required us to make some special commitment to a set of semantic values that are compatible with particular logics and with analysis. We preferred to adopt the transreal numbers as semantic values because we believe them to be compatible with all logics and we have proved, elsewhere, that they are compatible with analysis [33]. We were then obliged to develop the mathematics of trans-Cartesian and transvector spaces, trans-Boolean algebras, and to generalise the usual proofs of hypercyclicity. This established our main results about universal possible worlds and it gives us a mathematical foundation for many different logical and mathematical treatments, some of which we now discuss.

Having established trans-Boolean algebras, we may take the axes or points of our logical space as trans-Boolean terms and we may go on to construct higher cardinality spaces where we take the axes or points as terms. For example, in [16], we operate on the set of all possible worlds as a single term.

In his book, *Counterfactuals*, [24], David Lewis introduces the idea that for any possible world there are concentric spheres of possible worlds around it, whose similarity, to the central world, can be measured by their radius. This notion is vague but we can now put it on a firm, mathematical footing.

Recall that we have supplied a topological measure of distance, a particular metric, and we have indicated the existence of infinitely many Urysohn metrics [33]. Any of these is sufficient to order a transvector with a finite number of co-ordinates so, in general, we may consider the distance between worlds over some arbitrarily large, but fixed, number of co-ordinates. These co-ordinates relate to whatever atomic predicates are of interest to us but, conversely, we cannot distinguish between worlds that differ only in co-ordinates that we are not considering. This agrees with Lewis's view that there may be worlds that are effectively indistinguishable to us [24] p 15.

However, when transvectors lie in certain special configurations, we can usefully compute our distance measures over infinitely many co-ordinates. One such configuration is the comparison of a possible world with itself. If any of our distance measures are taken over infinitely many co-ordinates then the distance between two worlds is zero if and only if the worlds are identical. In other words, every world is identical only to itself, thus preserving identity across all possible worlds. This settles Lewis's doubt on this point, [24] p 14-15, 29.

In this paper we work with a finite distance metric $d(x, y)$, defined in equation 1. But we may choose to work with a transmetric [2]. Wherever we do work with a transmetric we may define similarity to be the reciprocal of distance. With a transmetric, worlds that are distant to a degree x are similar to a degree $1/x$. In particular, worlds that are identical have a distance apart of zero and a similarity of infinity because, in transreal arithmetic, $1/0 = \infty$; infinitely distant worlds have a similarity of zero because $1/\infty = 0$; and gap worlds, with a distance of nullity, are similar to degree nullity because $1/\Phi = \Phi$. Distance measures may be chosen to supply special weightings or to satisfy mathematical aesthetics. We expect that many readers will be content to use the trans-Euclidean metric [2], computed over finitely many, chosen co-ordinates. This metric, $t(x, y)$, returns zero for all equal arguments, $x = y$, and otherwise computes the Euclidean metric, $t(x, y) = \sqrt{(x - y)^2}$, but evaluated in transreal arithmetic.

Lewis proves [24] p 20, using an appeal to the continuum, that there is no distinct world that lies nearest to the central world. As the transreal numbers form a continuum, this result also holds in our logical space. However, Lewis also considers logical systems, which obey his *Limit Assumption* [24] p 19-20, in which there is a nearest, distinct world. Lewis obtains these worlds, non-paradoxically, by considering only the accessible worlds, [24] p 4-8, about a central world, not all possible worlds about a central world. In other words, Lewis punctures some possible worlds from the space of all possible worlds. This is a legitimate manoeuvre in our logical space. It jeopardises our hypercyclicity results but does not, necessarily, negate them. We have generalised hypercyclicity to a continuous, transreal space but hypercyclicity is also ordinarily established in various infinite lattices, in which case one might choose a puncturing that does not disturb a particular lattice, thus obtaining both Lewis's results in conditional, counterfactual logics and our results in hypercyclic, universal worlds. Alternatively one could accept that our hypercyclic results hold everywhere in a continuous space of all possible worlds and that Lewis's results hold in a subspace of this space.

Lewis considers centred logical systems in which the central world is present in a sphere and uncentered logical systems in which the central world is not present in the sphere [24] p 14. This latter is simply a puncturing of our logical space and is entirely unproblematic. Our hypercyclicity results are obtained by establishing limits to a central world and these limits need not arrive, indeed generally do not arrive, exactly at the central world.

Lewis considers that there is a continuum of concentric spheres that fill out a ball centred on any world [24] p 7. As our transreal space is continuous, it is composed of such solid balls.

Lewis considers asymmetric measures of distance [24] p 9, 50-52. These may be had by taking two or more copies of our logical space and applying different distance measures in each copy, for example by applying a single distance measure over different co-ordinates. Thus all of our results hold, relative to any particular choice of trans-Cartesian co-ordinate frame, set of co-ordinates, and choice of distance measure.

There is a trivial way to deal with conditional and, specifically, counterfactual implications: simply form the ordinary implication in every possible world then the conditional implication holds just in those world in which the ordinary implication holds and the conditional implication does not hold in all of the other worlds. Hence, by keeping track of subspaces, we can pursue even alternating sequences of non-monotonic reasoning to arbitrary degree.

Let us now turn our attention to some philosophical consequence of hypercyclicity. In topological terms a hypercyclic vector has a denumerable infinitude of components. A mechanical procedure, we use the backward shift operator, applied to a hypercyclic vector, produces a new, generally distinct, vector that lies in the orbit of its hypercyclic generator. The mechanical procedure may be applied a denumerable infinitude of times, in which case the orbit is infinitely dense in the whole of space. Furthermore, worlds in the orbit can be selected in sequences which converge to any particular world. If we regard a hypercyclic vector as a theory, say by virtue of being a binding of all possible atomic predicates to semantic values, then a mechanical procedure generates theories that are close to every world and we may refine the procedure by constructing a sequence of theories that converges to, but does not necessarily arrive at, an exact description of any particular world. This has an implication for the philosophy of science. Our actual world can be described, arbitrarily closely, by an infinitude of possible worlds. It is conceivable that human minds, in our actual world, might access such a hypercyclic world, by constructing a description of it, and go on, by either mechanical or more directed means, to discover everything to arbitrary accuracy. It has been our pleasure to supply an existential proof of this possibility but, regrettably, we are unable to provide a constructive proof!

6 Conclusion

We develop a numerical model for a total semantics and a geometrical model of the space of all possible worlds. We define the set of semantic values as the set of transreal numbers, $\mathbb{R}^T$, and define logical connectives in $\mathbb{R}^T$ so that we obtain a total semantics, that is, a semantics that contains all of the classical, paraconsistent, fuzzy and gap values. We define each possible world as a sequence of transreal numbers. We define trans-Boolean logic and transvector spaces and define accessibility relations between possible worlds as the existence of suitable, continuous, translinear transformations in the transvector space $(\mathbb{R}^T)^{\mathbb{N}}$. We show that this accessibility relation is reflexive and transitive. We show that the set of possible worlds, $(\mathbb{R}^T)^{\mathbb{N}}$, is a topological metric space and that there is a hypercyclic operator on it. This means that there are infinitely many possible worlds which can access any other by topological approximation. That is, we prove the existence of infinitely many universal, possible worlds.

We also observe that there are at least a denumerable infinitude of hypercyclic worlds that lie arbitrarily close to, and one of which may be identical with, the possible world that is an exact description of the real world we live in. This raises the possibility that entities within our world, such as human minds, might access one or more of these hypercyclic worlds and, thereby, embark on an unending process of discovering everything.

Acknowledgment

The authors would like to thank the members of Transmathematica for many helpful discussions.

References

[1] J. A. D. W. Anderson. Representing geometrical knowledge. Phil. Trans. Roy. Soc. Lond. Series B., 352(1358):1129–1139, 1997.

[2] J. A. D. W. Anderson. Perspex machine ix: Transreal analysis. In Longin Jan Lateki, David M. Mount, and Angela Y. Wu, editors, Vision Geometry XV, volume 6499 of Proceedings of SPIE, pages J1–J12, 2007.

[3] J. A. D. W. Anderson and W. Gomide. Transreal arithmetic as a consistent basis for paraconsistent logics. In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2014, WCECS 2014, 22-24 October, 2014, San Francisco, USA., volume 1, pages 103–108, 2014.

[4] J. A. D. W. Anderson and T. S. dos Reis. Transreal limits expose category errors in ieee 754 floating-point arithmetic and in mathematics. In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2014, WCECS 2014, 22-24 October, 2014, San Francisco, USA., volume 1, pages 86–91, 2014.

[5] J. A. D. W. Anderson, N. Völker and A. A. Adams. Perspex machine viii: Axioms of transreal arithmetic. In Longin Jan Lateki, David M. Mount, and Angela Y. Wu, editors, Vision Geometry XV, volume 6499 of Proceedings of SPIE, pages 2.1–2.12, 2007.

[6] F. Beyart and E. Matheron. Dynamics of Linear Operators. Cambridge University Press, 2009.

[7] J.-Y.Béziau. Frontiers of paraconsistent logic, chapter What is paraconsistent logic?, pages 95–111. Research Studies Press, 2000.

[8] G. D. Birkhoff. Démonstration d'un théorème élémentaire sur les functions entières. C. R. Acad. Sci. Paris, 189:473–475, 1929.

[9] J. Bonet and A. Peris. Hypercyclic operators on non-normable fréchet spaces. Journal of Functional Analysis, 159:587–595, 1998.

[10] J. P. Burgess. Saul Kripke, chapter Kripke Models. Cambridge University Press, 2011.

[11] W. Carnelli and J. Marcos. Ex contradictione non sequitur quodlibet. In Proc. 2nd Conf. on Reasoning and Logic, Bucharest, 2001.

[12] N. C. A. da Costa. The philosophical import of paraconsistent logic. Journal of Non-Classical Logic, 1:1–19, 1982.

[13] J. Floyd. Wittgenstein on philosophy of logic and mathematics. In Stewart Shapiro, editor, Oxford Handbook of Philosophy of Logic and Mathematics, chapter 4, pages 75–128. Oxford University Press, 2005.

[14] B. C. van Fraasen. Singular terms, truth values, gaps, and Frege logic. Journal of Philosophy, 63:481–495, 1966.

[15] G. Frege. Über Sinn und Bedeutung. Zeitschrift für Philosophie und philosophische Kritik, 100:25–50, 1892.

[16] W. Gomide, T. S. dos Reis and J. A. D. W. Anderson. Transreal logical space of all propositions. In Transactions on Engineering Technologies – World Congress on Engineering and Computer Science 2014. Springer-Verlag, 2015.

[17] K.-G. Grosse-Erdmann and A. Peris. Linear Chaos. Springer, 2011.

[18] P. Hajek. Metamathematics of Fuzzy Logic. Kluwer, 1998.

[19] T. M. V. Jansen. Frege, contextuality and composionality. Journal of Logic, language, and Information, 10, 2001.

[20] S. Kripke. Semantical analysis of modal logic. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 9:67–96, 1963.

[21] S. Kripke. Semantical considerations on modal logic. Acta Philosophica Fennica, 16:83–94, 1963.

[22] G. W. Leibniz. Monadology (originally titled la monadologie). In R. Franks and R.S. Woolhouse, editors, G.W. Leibniz Philosophical Texts. Oxford Philosophical Texts, 1998.

[23] G. W. Leibniz. Theodicy Essays on the Goodness of God, the Freedom of Man and the Origin of Evil. Gutenberg, 2005.

[24] D. Lewis. Counterfactuals. Blackwell, 1973.

[25] G. R. MacLane. Sequences of derivatives and normal families. Journal d'Analyse Mathématique, 2:72–87, 1952.

[26] P. Martin-Löf. On the meanings of the logical constants and the justification of the logical laws. Nordic Journal of Philosophical Logic, 1:11–60, 1996.

[27] J. R. Munkres. Topology. Prentice Hall, 2000.

[28] P. Petersen. Linear Algebra. Springer, 2012.

[29] G. Priest. Logic of paradox. Jounal of Philosophical Logic, 8(1):219–241, 1979.

[30] G. Priest. In Contradiction. A Study of the Transconsistent. Clarendon Press, 2006.

[31] G. Priest, J. C. Beall and B. A. Gab. The Law of Non-Contradiction. Oxford University Press, 2004.

[32] G. Priest, K. Tanaka and Z. Weber. Paraconsistent logic. The Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/archives/fall2013/entries/logic-paraconsistent, 2013.

[33] T. S. dos Reis and J. A. D. W. Anderson. Transreal calculus. IAENG International Journal of Applied Mathematics, 45(1):51–63, 2015.

[34] T. S. dos Reis and J. A. D. W. Anderson. Transreal limits and elementary functions. In Haeng Kon Kim; Mahyar A. Amouzegar; Sio long Ao, editor, Transactions on Engineering Technologies – World Congress on Engineering and Computer Science 2014, pages 209–225. SpringerVerlag, 2015.

[35] T. S. dos Reis, W. Gomide and J. A. D. W. Anderson. Construction of the transreal numbers and algebraic transfields. IAENG International Journal of Applied Mathematics, 46(1):11–23, 2016.

[36] W. Rudin. Functional Analysis. McGraw-Hill, 1991.

[37] B. Russel. Mathematical logic as based on theory of types. American Journal of Mathematics, 30:222–262, 1908.

[38] Y. Shramko and H. Wansing. Entailment relations and truth values. Bulletin of the Section of Logic, 36:131–143, 2007.

[39] P. Strawson. Identifying reference and truth values. Theoria, 30(2):96–118, 1964.

[40] A. Tarski. Introduction to Logic and to the Methodology of Deductive Sciences. Dover Books, 1940.

[41] L. Wittgenstein. Logisch-philosophische abhandlung. Unesma, 1921.

[42] L. Wittgenstein. Tractatus Logico-Philasophicus. Gutenberg, 2010.

[43] L. A. Zadeh. Fuzzy logic and approximate reasoning. Synthese, 30:407–428, 1975.

Walter Gomide
Department of Philosophy
Institute of Humanities and Social Sciences
Federal University of Mato Grosso
78060-900
Brazil
E-mail: `waltergomide@yahoo.com`

Tiago S. dos Reis
Federal Institute of Education
Science and Technology of Rio de Janeiro
Rio de Janeiro
27215-350
Brazil
E-mail: `tiago.reis@ifrj.edu.br`

James A. D. W. Anderson
Department of Computer Science
School of Mathematical, Physical and Computational Sciences
University of Reading
England
RG6 6AY
E-mail: `j.anderson@reading.ac.uk`